Depth-averaged Two-phase Debris-flow Model with a Shear-rate Dependent Internal Friction

Xie Y.X., Zhou G.D., Song D.R., Cui, K.F., Zhou M.J.

a Key Laboratory of Mountain Hazards and Earth Surface Process, Chinese Academy of Sciences, Chengdu, China

b Institute of Mountain Hazards and Environment, Chinese Academy of Sciences & Ministry of Water Conservancy, Chengdu, China

ABSTRACT

The dynamics of debris flows are highly dependent on the interaction between its solid and fluid constituents. To capture the additive effects of the individual dynamics of each phase to the overall mobility, a series of two-phase depth-averaged equations are proposed. The fluid phase is modelled as a Newtonian fluid while the solid phase is modelled as a granular flow with a contact friction law that is based on a modified non-local $\mu(I)$ rheology. This rheological model features a shear-rate dependent internal friction with input parameters that include the solid volume fraction, the mean particle diameter, and the coefficients of friction for quasi-static and rapid granular flows. The governing equations are solved using a non-staggered central scheme that is based on the central finite difference method. The proposed model was used to simulate the dynamics of the Yu Tung channelized debris flow of Hong Kong. It was found that the presence of interstitial fluid greatly affected the modelled flow velocities, and a relatively high fluid content (lower solid volume concentration) is necessary to generate values that are in good agreement with field measured data.

1 INTRODUCTION

Debris flows are characterized by their lack of characteristic grain size and the abundance of interstitial pore fluid (Iverson & George 2001). Constituent particles can be as large as stones and boulders or can be as fine as silt and clay. The dynamics of debris flows are largely driven by the fluctuations and interactions of the larger particles. The fine particles mix with the pore fluid and effectively become part of the fluid phase, forming a more viscous muddy suspension. The re-arrangement of particles generate transient excess pore water pressures and their relative velocities with respect to the fluid affects the velocity of the fluid itself. At the same time, the fluid provides a buoyant force which reduces the amount of energy lost due to particle collisions and hence enhances the over-all mobility of the flow. Needless to say, what makes the dynamics of debris flows unique compared to other geophysical mass flows is the coupling, and not only the additive effects, of both its solid and fluid phases (Iverson 1999). In modelling debris flows and other related phenomena (e.g. mudflows, debris avalanches, etc.) it is therefore important to consider the effects solid-fluid interactions.

An additional consideration in debris flow modelling is choosing the appropriate rheological laws that would enable us to correctly simulate the physics involved in such systems. Commonly, the Coulomb and Voellmy friction models are used to define the basal contact friction of granular flows. Coulomb models only apply to quasi-static conditions, treating the system as a solid block and are therefore unable to capture the effects of particle contacts within the granular body. The Voellmy model performs relatively well against the Coulomb model, as it adds a dynamic shear resistance term expressed as a function of an empirical turbulence coefficient (Hungr & Evans, 1997; Crosta et al. 2004). It still suffers from a fundamental problem, however, as this turbulence coefficient is not physically well-defined. To account for the contact friction, shear rate dependency, and channel confinement, a modified $\mu(I)$ friction model (Jop et al. 2006) based on the non-local rheology of granular flows was developed (Zhou et al. 2013). In this model the shear stress is a function of the normal stress and the internal friction which in itself varies with the shear rate. The shear rate dependence enables the frictional model to transition from a quasi-static regime to a rapid shear regime allowing it to capture a wider range of the flow’s dynamics at an improved accuracy.
In this paper we present: (i) a hierarchical set of depth averaged equations for the motion of a two-phased continuum which incorporates the non-local $\mu(I)$ frictional model, and (ii) a numerical scheme to discretize and solve the set of equations and the frictional model. The applicability of the developed model was tested on the Yu Tung Debris Flow in Hong Kong whose data and details were provided by the Geotechnical Engineering Office of Hong Kong (GEO Report No. 271). The test case was chosen as it was the closest in relevance to the developed model. The rest of this report is structured as follows: first, the governing equations for dynamic granular flows, along with the featured rheological model and the numerical scheme used are presented and briefly explained. This is then followed by the results of the benchmarking exercise done on the case of choice. Finally, the advantages and limitations of the presented model are discussed.

2 MATHEMATICAL MODEL

2.1 Two-phase Dynamic Model for Granular Flows

The two-phase continuum model considering the shear rate dependent contact friction is adopted from the conservation of mass and momentum:

\[
\begin{align*}
\rho_s (\partial_t (Cs) + \nabla \cdot (Cs v_s)) &= \Delta m_s \\
\rho_f (\partial_t (1 - Cs) + \nabla \cdot (1 - Cs)) &= \Delta m_f \\
\rho_s Cs (\partial_t (v_s) + \nabla (v_s \cdot v_s)) &= -\nabla \cdot T_s + f_t + \rho_s Csg \\
\rho_f (1 - Cs) (\partial_t (v_f) + \nabla (v_f \cdot v_f)) &= -\nabla \cdot T_f - f_t + \rho_f (1 - Cs) g
\end{align*}
\]  

(1)

where $\rho$ is density, $g$ is the gravitational acceleration, $\nu$ is the velocity vector and $T$ is the stress tensor. The subtitile $s$ and $f$ represent the solid and fluid phase, respectively. For most geophysical mass flows, the flow length and width greatly exceed its flow depth $h$ (Savage & Hutter, 1989). These systems can be regarded as shallow water flows where the depth averaged equations can be used to define the corresponding dynamics. Therefore, the following equations of hyperbolic conservation laws which are derived from the conservation of mass and momentum equations are proposed:

\[
\begin{align*}
\frac{\partial (h_s)}{\partial t} + \frac{\partial (h_s v_s)}{\partial x} + \frac{\partial (h_s u_s)}{\partial y} &= 0 \\
\frac{\partial (h_f)}{\partial t} + \frac{\partial (h_f v_f)}{\partial x} + \frac{\partial (h_f u_f)}{\partial y} &= 0 \\
Cs &= \frac{h_s}{h_f + h_s} \quad \rho_m = \rho_s Cs + \rho_f (1 - Cs); v_m = \frac{\rho_s Csv_s + \rho_f (1 - Cs)v_f}{\rho_s Cs + \rho_f (1 - Cs)} \\
\rho_s \left( \frac{\partial (h_s u_s)}{\partial t} + \frac{\partial (h_s u_s^2 + k_s gh_s^2/2)}{\partial x} + \frac{\partial (h_s u_s v_s)}{\partial y} \right) &= \rho_s gh_s \frac{\partial (z + h_s + h_f)}{\partial x} - T_{sx} + f_{ix} \\
\rho_s \left( \frac{\partial (h_s v_s)}{\partial t} + \frac{\partial (h_s v_s u_s)}{\partial x} + \frac{\partial (h_s u_s^2 + k_s gh_s^2/2)}{\partial y} \right) &= \rho_s gh_s \frac{\partial (z + h_s + h_f)}{\partial x} - T_{sy} + f_{iy} \\
\rho_f \left( \frac{\partial (h_f u_f)}{\partial t} + \frac{\partial (h_f u_f^2 + k_f gh_f^2/2)}{\partial x} + \frac{\partial (h_f u_f v_f)}{\partial y} \right) &= \rho_f gh_f \frac{\partial (z + h_s + h_f)}{\partial x} - T_{fx} - f_{ix} \\
\rho_f \left( \frac{\partial (h_f v_f)}{\partial t} + \frac{\partial (h_f v_f u_f)}{\partial x} + \frac{\partial (h_f u_f^2 + k_s gh_f^2/2)}{\partial y} \right) &= \rho_f gh_f \frac{\partial (z + h_s + h_f)}{\partial x} - T_{fy} + f_{iy}
\end{align*}
\]  

(2)

(3)

(4)

(5)

For convenience, the following set of equations can be re-written in vector form:
\[
U = \begin{pmatrix}
h_s \\
h_s u_s \\
h_s v_s \\
h_f u_f \\
h_f v_f
\end{pmatrix},
F(U) = \begin{pmatrix}
h_s u_s \\
h_s u_s^2 + k_s g h_s^2 / 2 \\
h_s u_s v_s \\
h_f u_f \\
h_f u_f^2 + k_f g h_f^2 / 2
\end{pmatrix},
G(U) = \begin{pmatrix}
h_s u_s \\
h_s u_s v_s \\
h_s v_s^2 + k_s g h_s^2 / 2 \\
h_f u_f \\
h_f v_f^2 + k_f g h_f^2 / 2
\end{pmatrix},
\]

where \( h, h_u, \) and \( h_v \) represent the thickness from the base to the free surface, and the discharge per unit width in the \( x, y \) directions respectively; The subtitle \( s \) and \( f \) represent the solid and fluid phase, respectively; \( C_s \) is the solid volume fraction; \( g = (g_x, g_y, g_z) \) is the gravitational acceleration; \( u, v \) are the depth-averaged velocity components in the \( x, y \) directions; \( k_{ap} \) is the lateral earth pressure coefficient; \( f_i = (f_{ix}, f_{iy}) \) represents the particle-fluid interaction. The basal friction adopted for the liquid phase is simply the fluid viscosity while the solid part uses the shear-rate dependent, non-local \( \mu(I) \) rheology.

2.2 Non-local \( \mu(I) \) Rheology

The \( \mu(I) \) friction model is based on the rheology of granular flows which takes into account the contact friction, shear rate dependency, and effects of channel confinement on the flow behavior (MiDi, 2004). It expresses the shear stress as a function of the normal stress (which is primarily from the lithostatic pressure) and an internal friction coefficient. The friction coefficient is in turn a function of the inertial number \( I \), a dimensionless number which physically represents the ratio of two microscopic time scales: \( T_1 = \frac{a}{\sqrt{\rho / \rho_d}} \) and \( T_2 = \frac{1}{\gamma} \) (MiDi 2004).

The shear stress and the complete expression of \( \mu \) can be written as:

\[
\tau = P \cdot \mu(I)
\]

\[
\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0 + 1}
\]

where \( P \) is the normal stress acting on the basal surface; \( \mu_s \) and \( \mu_2 \), are the coefficients of friction for quasi-static and rapid flows respectively. \( I_0 \) is a solid particle constant. From eqn. (1), transition from a quasi-static regime to a rapid shear regime occurs when the shear rate or pressure changes (Forterre & Pouliquen, 2008). The values of \( I \) provide a quantitive representation of the state of motion / flow regime the particles within a granular flow belong to: \( I < 0.1 \) suggests a dense flow where particles have lasting contacts while \( I > 0.1 \) suggests that the flow is collision dominated where particles are highly energetic and undergo transient contacts.
In the new frictional model, the expressions for \( I \) and \( I_0 \) is modified to incorporate material properties with clear physical meanings:

\[
I \approx \frac{ud_{50}}{h^2 \sqrt{C_s g}}; \quad I_0 = \frac{5 \ d \ \beta}{2 \ l_0 \ \sqrt{\phi \cos \theta}}
\]

where \( \beta \) and \( l_0 \) are material constants; \( d_{50} \) is the mean particle diameter; \( h \) and \( u \) are the granular flow depth and velocity respectively. Considering the revised expressions in eqn. (5), the shear stress \( \tau \) can be rewritten as:

\[
\tau = \rho g h \left( \mu_s + \frac{(\mu_2 - \mu_s) ud_{50}}{l_0 h^2 \sqrt{C_s g} + ud_{50}} \right)
\]

2.3 Fluid-Solid Interaction force

The interaction force between fluid and solid particle consists of two parts: hydrostatic and hydrodynamic forces. The hydrostatic forces accounts for the fluid pressure gradient around the individual particle. The hydrodynamic forces acting on a particle are the drag, lift, and virtual mass forces. The latter two forces are normally very small when compared to the drag force in simulating fluid flow at relatively low Reynolds number (Kafui et al. 2002). Thus, the lift and virtual mass forces are neglected in the current coupling model.

\[
f_i = f_b + f_l + f_d \approx f_b + f_d
\]

where \( f_b = \nabla \rho \nabla P \) and \( f_l = \rho \nabla \phi \nabla P \)

The momentum transfer coefficient is derived from the experimental correlations as:

\[
\beta = \frac{3C_d(1 - n) \rho_f |U - V|}{4D}
\]

where \( C_d \) is the drag force coefficient; \( n \) is the porosity of soil; \( D \) is the particle diameter

\[
C_d = \begin{cases} 
\frac{24}{R_{ep}} & R_{ep} \leq 10^3 \\
(1.0 + 0.15 R_{ep}^{0.687}) R_{ep} & R_{ep} > 10^3 \\
0.44 & R_{ep} > 10^3 \\
(0.63 + 4.8 \sqrt{R_{ep}})^2 & R_{ep} > 10^3 \\
\frac{24}{R_{ep}} (1.0 + 0.15 R_{ep}^{0.681}) + \frac{0.407}{1 + 8710 / R_{ep}} & R_{ep} > 10^3 
\end{cases}
\]

The equations of (11) is obtained by experiments of fluidization on dense granular beds, it is not applicable for dilute granular sample. The equations of (12.2) would cause discontinuity in calculating drag forces if the granular sample porosity varies. To get rid of this problems, the drag force model proposed by Di Felice (1994) is used in the simulation. The drag force is defined as:

\[
f_d = \frac{1}{2} C_d \rho_f \frac{\pi D^2}{4} |U_s - U_f| (U_s - U_f) n^{-X+1}
\]

where \( n^{-X+1} \) is the porosity correction function which represent the influence of the concentration of granular materials on the drag force. The term \( X \) is defined as:

\[
X = 3.7 - 0.65 e^{-\frac{(15 - \log_{10} R_{ep})^2}{2}}
\]

2.4 Numerical Scheme
In this study, a new family of central schemes is adopted which can be treated as a natural extension of the Lax-Friedrich first-order scheme. The central Nessyahu—Tadmor (NT) scheme offers higher resolution while retaining the simplicity of the Riemann-solver-free-approach (Nessyahu & Tadmor 1990). The cells in the NT scheme alternate every adjacent time step $\Delta t$. The importance of staggering is due to the fact that cell interfaces are stable in neighborhoods around the smooth regular mid-cells of the previous time step. The use of this central difference scheme is attributed to the advantage of avoiding the costly Riemann characteristic decomposition as well as retaining high resolution. The staggered form is then converted into a non-staggered form by re-averaging the reconstructed value of the staggered grid, which helps to overcome some inconvenience near the boundaries, and the discomfort of having the values change due to the alternate cells at every adjacent time step $\Delta t$.

3 BENCHMARK RESULTS

The Yu Tung Debris Flow of Hong Kong is a series of 19 landslides that occurred at the hillside area adjacent the Yu Tung road. The events were reported to be triggered by extreme and prolonged rainfall. Four of them developed into 3 channelized debris flows which resulted to flooding and serious road blockage. For the purposes of this benchmark, the largest of debris flow events was singled out. This event had a total detached mass of 2,350 m$^3$ and a total run-out distance of 600m. Its maximum velocity was estimated to be about 12m/s at a distance of 100m from the source location which reduced to about 10m/s at 400 m. This debris flow was characterized as being very mobile whose mobility was attributed to the volume of its debris, drainage line morphology, and water content.

The event was simulated using the two-phase model with the parameters listed in table 1. The solid volume concentration $C_s$ is set low implying the presence and prevalence of interstitial fluid in the mixture. The interstitial fluid is assumed to be a slurry—a mixture formed from the suspension of very fine particles in water—which is why the fluid viscosity is set relatively high. The values of the rheological parameters $\mu_s$, $\mu_2$ and $I_0$ which are based from Zhou et al. (2013).

From the simulated velocity profiles for different solid volume concentrations $C_s$ (0.35 and 0.6) in Fig. 1b, it can be seen that generally the flow ramps up at initiation and then slowly increases up until a distance of 250m. It is at this distance along the flow path that the moving material reaches its peak velocity. After this, the flow then begins to slow down as it progresses further downwards until it reaches its deposition point at around 590m. The mixture with higher $C_s$ has lower flow velocities and a slightly shorter flow distance. This shows the effect of the presence of interstitial fluid in enhancing flow mobility.

The flow profiles and corresponding velocities at 100m, 400m, 420m and at deposition are further shown in Fig. 2 for $C_s = 0.35$. A certain degree of spreading is observed at the onset of the flow. As it progresses, it can be seen that the flow is highly channelized and leaves a trail of the material along the path. The flow velocity is more or less consistent up until it stops and deposits. The deposition area is observed to be a few meters less than that which is reported and doesn’t exactly reach the road.

The calculated velocities are compared with the actual velocities which are estimated at certain points along the flow path (obtained from the material provided in this benchmarking exercise). These points are labelled as A, B, C, D and E and are indicated in Figs. 1. For a $C_s = 0.35$, it can be seen that the velocities at points A, B, and C are calculated with good accuracy although a slight deviation is observed for the latter. Great deviations from the actual values are observed for points C and D. The velocities at these areas appear to be greater than the preceding values, approximately 20m away, implying that the flow is still accelerating despite the consistency of the flow path. The calculated velocities however only exhibit a continuous decline resulting in the observed underestimation. For higher solid concentrations ($C_s = 0.60$), no agreement is observed at all between the measured and simulated velocities. This confirms with recorded accounts that the said event involved relatively large amounts of pore fluid resulting from the exposure of the source area to heavy rainfall.

| Table 1: List of parameters and corresponding values used in the numerical model |
|---------------------------------|-------------|
| Solid Parameters               | Values      |
| Solid Volume Concentration $C_s$ (m) | 0.35  |
| $\mu_s$                        | 0.22       |

5
\[
\begin{array}{|c|c|}
\hline
\mu_2 & 0.3 \\
I_0 & 0.3 \\
\text{Mean Particle Diameter } d_{50} (m) & 0.01 \\
\hline
\end{array}
\]

### Fluid Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid Viscosity ( (\text{Pa} \cdot \text{s}) )</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**Fig. 1** a) Ortho-graphic projection of the channelized debris flow event no. 25 that happened adjacent to the Yu Tung road, Hong Kong (the image is taken from the material provided for this benchmarking exercise). The demarcations indicate the locations where flow velocities are estimated. b.) Velocity profiles as calculated from the two-phase model for \( C_s = 0.35 \) (dashed-dot), \( C_s = 0.6 \) and the elevation profile of the flow path (solid line). The points in the graph are the known velocities in certain regions as previously shown in a.

**Fig. 2** The flow profiles at 100m, 400m, 420m and at deposition for \( C_s = 0.35 \). The final image is the deposit profile which stops a few meters from the road.

### 3 CONCLUSIONS
Debris flows are characterized by the additive and coupled effects of its solid and fluid constituents to the flow dynamics. As such, a two-phase depth-averaged debris flow model is proposed which aims to simulate the motion of each constituent during a single event. The solid phase is treated as a dense granular flow with the frictional resistance defined by the shear-rate dependent non-local $\mu(I)$ rheology. The fluid phase is treated as a slurry made up of water and fine particles. Fluid friction is simply defined by the fluid viscosity.

The model was tested against a debris flow case which was known to be highly saturated and thus has a fluid phase that can affect the mobility significantly. The model parameters were chosen based on previously determined values. The solid volume concentration was varied to demonstrate the effect of the degree of saturation to the flow mobility. Increasing the solid concentration accordingly decreases the flow velocity and the deposit distance. This implies that the presence (or abundance) of pore fluid effectively reduces internal friction and hence numerically “fluidizes” the flow.

The model provides generally acceptable results but still suffers from several shortcomings. In this brief report, a comparison between other frictional models was not presented nor was an in-depth assessment of the numerical values used provided. Further investigation into these matters will be the focus of future work.

ACKNOWLEDGEMENTS

The authors acknowledge financial support from the Key Research Program of the Chinese Academy of Sciences (grant no.KZZD-EW-05-01).

REFERENCES


